LEVERED EQUITY RETURNS IN THE PRESENCE OF RISKY DEBT: AN APPLICATION TO PRIVATE REAL ESTATE

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Using a simple reduced-form model to approximate the pricing of risky debt, this article focuses on three aspects of levered equity returns when investing in private commercial real estate: First, the positive relationship between the project's leverage ratio and the costs of mortgage-loan borrowing creates a concave risk/return continuum for levered equity investors. The lender's pricing of risky debt is such that the equilibrium return on levered equity asymptotically reaches a maximum; while additional borrowing increases the expected return on equity, it only marginally does so. Second, the law of one price suggests that this curvilinear continuum represents the basis on which higher-risk/higher-return investments ought to be measured – as opposed to the more widely theorized linear relationship (which assumes constant borrowing costs). Third and when considering delegated investment management, additional borrowing linearly (or nearly so) increases the expected value of the sponsor's (or general partner's) promoted interest, which directly reduces the expected return of the investor's (or limited partner's) net return, creating a maximum leverage ratio for the investor which is less than that of the fund's. In almost all cases, this reduction in the investor's (net) return leads to an optimal leverage ratio well below the maximum available. All three of these impacts are attenuated at the highest leverage ratios. The balance of this paper examines these impacts.

I. The Pricing of Risky Debt

Given the prevalence with which financial leverage is used in private (and public) real estate investing, understanding the impacts of such leverage on the risk/return characteristics of levered equity is critical. This article focuses on the use of "risky" debt. Moreover, this article seeks to endogenously incorporate the cost of such debt when examining levered returns. It is well accepted both in theory and practice that the risk structure of interest rates (*i.e.*, the "credit spread") geometrically increases the cost of indebtedness as leverage increases (*i.e.*, interest rates are curvilinear (and convex) holding all other factors constant).[1] The commercial mortgage lender's instrument is often viewed as providing the (non-recourse) borrower with a put option[2] and the estimated pricing of this put option naturally leads to an application of the contingent-claims approach (*e.g.*, see Titman and Torous (1989) with regard to commercial mortgage debt). However,

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[1] One exception is the FHA/HUD lending program (*e.g.*, §221(d)(4), §223(a)(7) and §223(f)) which does not vary the interest rate either by the leverage ratio or by asset/borrower quality. As a result, the FHA/HUD lending program suffers two main effects: adverse selection and excessive leverage. See Pagliari (2012).

[2] The borrower's ability to "hand back the keys" without incurring further liability is a put option granted to the non-recourse borrower by the lender in return for a higher interest rate than would otherwise be the case. That is, the lender's payoff at loan maturity (T) – assuming no earlier default – equals the minimum of the asset's value (A_T) and the loan's book balance (D_T). In the parlance of option pricing, this payoff amount is often written as $\min(A_T, D_T)$, which is equivalent to $D_T - \max(D_T - A_T, 0)$; this last term signifies the borrower's put option. Sundaresan (2013) provides an extensive survey of the literature relating to the Merton (1974) model and the firm's capital structure.

the contingent-claims approach, as initiated by Black and Scholes (1973) and Merton (1974), fails to fit the conditions of private borrowings in three material respects: [3]

1. The perfect, costless and continuously available hedges – such that the risk-free portfolio can be replicated – found (or, at least, nearly so) in the public markets are not found in the private markets. As such, the no-arbitrage arguments used to price risky debt fail to hold.

2. While the risk/return performance of the underlying security is exogenous to the efforts of the option holder in the public marker, that performance is endogenous in the private market (*i.e.*, the borrower's efforts and risk-taking effect the performance of the collateral). Therefore, the issues of "financial distress" (*i.e.*, not only the deadweight costs of bankruptcy, but also the issues of effort, "risk-shifting," "asset substitution," *etc.*[4]) are particularly problematic.[5] Consequently, the lender sets some upper limit on the amount of permitted leverage which is well below the near 100% contemplated (in the limit) in Merton (1974). While it is possible that an estimate of these costs is incorporated into an option-pricing exercise, the imprecision of the estimate belies the exactitude of the option-pricing exercise.

3. In a bankruptcy/foreclosure proceeding, the lender's claim may be restructured by the decree of the bankruptcy judge. This "cram down" (which may involve a reduction in the interest rate, a lengthening of the amortization schedule, an extension of the loan's maturity date, *etc.*) may severely and adversely impact the fair market value of the lender's claim. This external influence is altogether different from the environment of public-market options – in which, there is no restructuring of the claim upon exercising the option.

Moreover, it is well known that the Merton (1974) model tends to understate actual credit spreads for corporate bonds (*e.g.*, see: Huang and Huang (2012)). Culp, Nozawa and Veronesi (2018) indicate that a risk premium for tail and idiosyncratic (asset) risks is the primary determinant of this understatement and debunk several other reasons (*e.g.*, illiquidity, ruthless corporate defaults, large bankruptcy costs, asymmetric information, *etc.*) for this understatement. Therefore, rather than

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[3] Additionally, neither Merton (1974) nor this paper pay sufficient attention to the path-dependent nature of these loans. In order for the borrower to default upon the mortgage loan, two conditions must be satisfied: *a)* the loan's debt service exceeds the property's cash flow (and the owner is unwilling/unable to fund the difference) and *b)* the loan balance exceeds the property's fair market value (less transaction costs). Modeling these compound options is complicated by the autoregressive nature of private-market asset returns and the specific covenants (*e.g.*, amortizing *v.* interest-only) of the mortgage loan.

[4] It is quite possible that the borrower's view of financial distress is broader than the lender's. For the borrower, distress may include the need to issue additional equity which is dilutive to the initial equity in order to recapitalize a poorly performing asset.

[5] Holmstrom and Tirole (1997) point out that a credit crunch hits the poorly capitalized firm the hardest. This heightens the lender's attention to the maximum leverage ratio.

employing the option-pricing apparatus, let's propose a simpler, more-tractable function of the lender's pricing (k_d) of risky debt; importantly, such an approach will permit us to easily treat financing costs as endogenous to the borrower's leverage decision – as opposed to the more traditional treatment which often assumes exogenous, constant debt cost. This reduced-form function requires several salient features; it should: *a*) be convex (with regard to the leverage ratio), *b*) have a maximum leverage ratio – given the concerns indicated above – that is well below 100%, *c*) have a boundary condition that approaches the mortgaged asset's expected return [$E(k_a)$], and *d*) be tied to the volatility (σ_a) of the mortgaged asset's return. Specifically, let's assume that this pricing function can be approximated by:

$$k_d = r_f + \gamma + \delta \frac{LTV}{1 - LTV} \tag{1}$$

where: r_f = the risk-free (*e.g.,* Treasury) rate, γ = structural differences as between the Treasury and commercial mortgage markets,[6] δ = the price of default risk, and LTV = the leverage ratio (*i.e.*, the debt amount relative to asset value).[7] Equation (1) is clearly convex[8] (with regard to the leverage ratio), as it produces a risk-pricing structure that can be illustrated as in Exhibit 1:

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[6] More completely, these differences include:
- Treasury bonds make semi-annual interest-only coupon payments, while mortgage loans generally make monthly payments of principal and interest. (As such, the durations of these two securities differ when they have identical maturities.)
- Third-party servicing fees are deducted from the mortgage loan payments prior to investors receiving their cash flow. There may also be additional internal costs to monitor the mortgage-loan collateral (which is not the case for U.S. Treasuries).
- Because of the inherent lag between the dates of the mortgage loan commitment and the loan funding, lenders often hedge the risk of (fixed) interest-rate movements with interest-rate swaps. (These hedging costs are specific to the mortgage loans.)
- While Treasury bonds and (commercial) mortgage loans are non-callable, the risky nature of the mortgage loans however suggests that investors face re-investment rate risk which cannot be contracted away (either by yield-maintenance or defeasance provisions) in the face of default risk.

[7] This paper takes no stand on whether lenders decide to securitize their mortgages or retain them on their balance sheets (*e.g.*, see: Ambrose, Lacour-Little and Sanders (2005)).

[8] A comparison of Equation (1) to the more traditional option-pricing approach is found in Appendix A.

Exhibit 1: Illustration of the Cost of Indebtedness as a Function of the Leverage Ratio
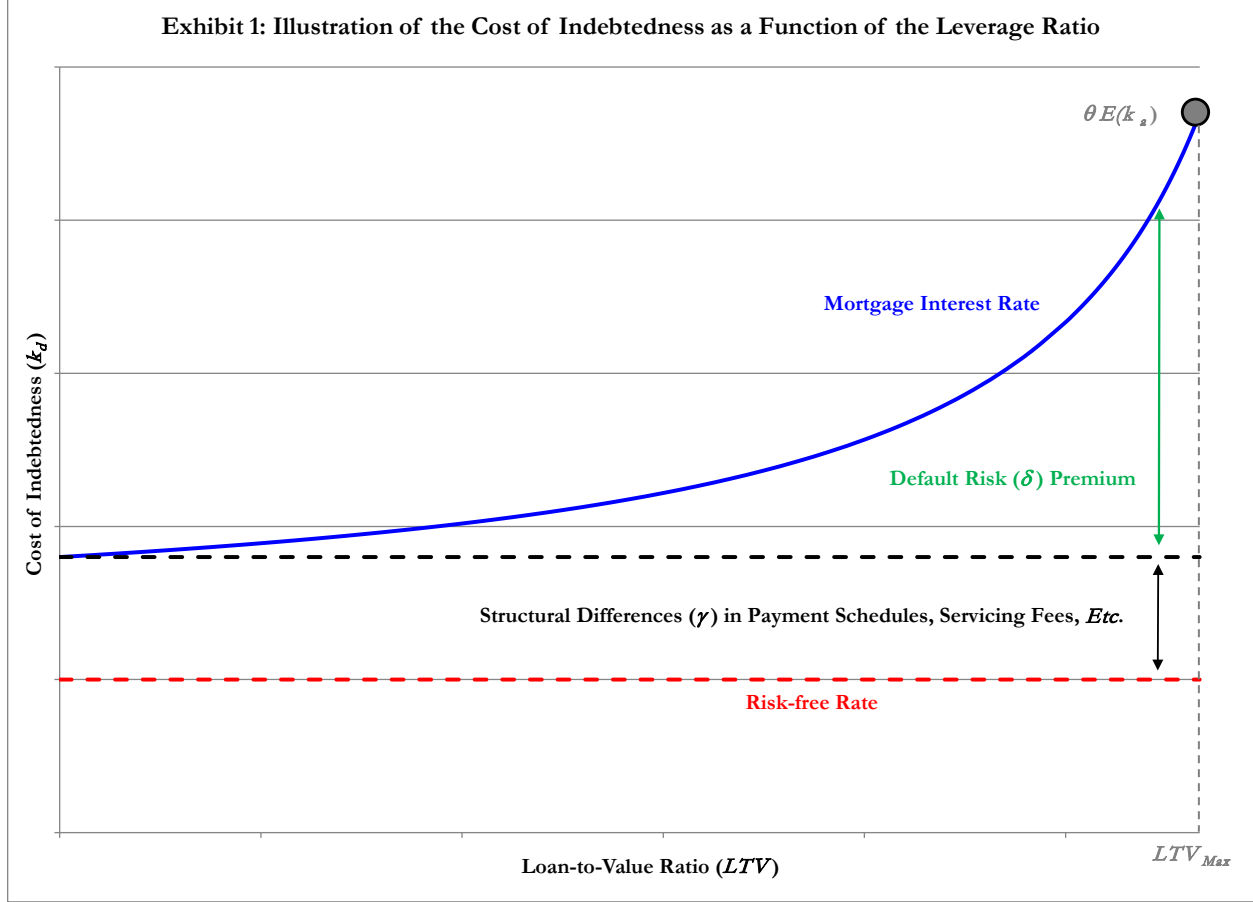
Let's further designate a maximum leverage ratio[9] ($.5 < LTV_{Max} < 1$) which is well below 100%, thereby satisfying the lender's earlier-mentioned concerns about the borrower's effort, financial distress, risk shifting, *etc.* Let's additionally stipulate that, at this boundary condition, the cost of indebtedness equals a percentage, θ, of the mortgaged asset's (or collateral's) expected return [$E(k_a)$]. Then in the limit, as $LTV \rightarrow LTV_{Max} \Rightarrow k_d \rightarrow \theta * E(k_a)$.

Consequently, we can invert Equation (1) to solve for the pricing (δ) of risky debt using this boundary condition at LTV_{Max}:[10]

$$k_{d|LTV_{Max}} = r_f + \gamma + \delta \frac{LTV_{Max}}{1 - LTV_{Max}} = \theta E(k_a)$$

(2)

$$\delta = \left[\theta E(k_a) - r_f - \gamma \right] \frac{1 - LTV_{Max}}{LTV_{Max}}$$

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[9] Given our setup, LTV_{Max} must be greater than 50% for technical reasons.

[10] In equilibrium, $\theta E(k_a) > r_f + \gamma$ and, accordingly, δ is always positive.

Notice that the pricing (δ) of default risk is a function of the asset's expected return (as well as: r_f, γ and LTV_{Max}).[11] In equilibrium, an asset's expected return is tied to the volatility (σ_a) of that asset's return and, as such, the default premium is also a function of the asset-level volatility.[12]

Therefore, all four of the earlier-mentioned salient requirements (*i.e.*, convexity, maximum leverage ratio, boundary condition, and volatility-based pricing) are satisfied by Equation (1).

II. <u>Levered Equity Returns with Risky Debt</u>

Let's next turn our attention to the expected return on levered equity [$E(k_e)$]. To do so, let's use a simple restatement of the one-period model of levered returns of Modigliani and Miller (1958), augmented by the inclusion of risky debt (as denoted in Equation (1)) and thereby making debt costs endogenous to the borrower's leverage decision:

$$E(k_e) = \frac{E(k_a) - k_d LTV}{1 - LTV}$$

(3)

$$= \frac{E(k_a) - \left(r_f + \gamma + \delta \dfrac{LTV}{1 - LTV} \right) LTV}{1 - LTV}$$

Without loss of generality, let's presume fixed-rate financing; in such a financing environment, the

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[11] Another interesting difference as between the reduced-form approach suggested here and the more-traditional option-pricing approach used elsewhere is the emphasis on $E(k_a)$ here (or $\theta E(k_a)$, more accurately) as compared to the emphasis on σ_a elsewhere (in both cases, holding constant other factors). That is, this paper argues that $k_d \rightarrow \theta E(k_a)$ while option-pricing models use σ_a to price the borrower's put option. In certain states of the world in which $E(k_a)$ is low and σ_a is high, it is possible that the option-pricing approach produces $k_d > E(k_a) > \theta E(k_a)$. In one sense, this clearly violates the equilibrium condition that the borrower cannot plausibly promise to pay the lender more than the $E(k_a)$. On the other hand, it may well benefit the lender in this state of the world ($E(k_a) \downarrow$ and $\sigma_a \uparrow$) to quote credit spreads greater than is justified based on expectations about future asset-level returns.

[12] In equilibrium, the mortgaged asset's expected return [$E(k_a)$] is a function, as determined by the marketplace, of the volatility (σ_a) of the asset's return. Consequently, both δ and θ are also functions of asset volatility – because they are functions of the asset's expected return. As an example, it is widely believed that the volatility of returns from hotel properties exceeds that of industrial properties ($\sigma_{a|Hotels} > \sigma_{a|Industrial}$) and, accordingly, the expected returns from hotel properties exceeds that of industrial properties [($E(k_{a|Hotels}) > (E(k_{a|Industrial})$]. Consequently, the price of default risk is higher for hotels than it is for industrial properties ($\delta_{Hotels} > \delta_{Industrial}$). And by similar reasoning, one would expect that the maximum leverage ratio also varies by the collateral's risk attributes (*e.g.*, $LTV_{Max | Hotels} < LTV_{Max| Industrial}$). Accordingly, the mortgage interest rate and the risk terms are jointly determined. See Donaldson and Wetzel (2018), which indicate that mortgage contracts are equilibrium outcomes of a multidimensional negotiation between borrower and lender.

volatility of levered equity returns (σ_e) is:[13]

$$\sigma_e = \frac{\sigma_a}{1 - LTV} \tag{4}$$

where σ_a = the volatility of (unlevered) asset returns. Additionally, Equation (4) can be substituted into Equation (3), such that the expected return on levered equity is restated in terms of volatility. When so doing, Equation (3) becomes:

$$E(k_e) = r_f + \gamma - \delta + \frac{\left(E(k_a) - r_f - \gamma + 2\delta\right)}{\sigma_a}(\sigma_e) - \frac{\delta}{\sigma_a^2}(\sigma_e^2) \tag{5}$$

From this restated version of the expected return on levered equity, Equation (5) makes clear that the risk/return continuum is a positive function of the standard deviation of levered equity returns and a negative function of its variance.[14]

To help fix these ideas, let's parameterize Equations (1) – (5) as follows: r_f = 2.5%, γ = 0.4%, LTV_{Max} = 90%, $E(k_a)$ = 7.5% and σ_a = 10%.[15] Additionally, equilibrium requires that $\frac{r_f + \gamma}{E(k_a)} < \theta < 1$; so, let's consider three potential values (or ranges) of θ. Accordingly, the risk/return continuum of levered equity, in the presence of risky debt, can be illustrated as in Exhibit 2.

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[13] Equation (4) derives algebraically from Equation (3), when fixed-rate financing is used and such debt is not marked to market. If floating-rate debt is used (or fixed-rate debt is regularly marked to market), then the volatility of levered equity returns is given by: $\sigma_e = \sqrt{\left(\frac{1}{1-LTV}\right)^2 \sigma_a^2 + \left(\frac{LTV}{1-LTV}\right)^2 \sigma_d^2 - 2\frac{LTV}{(1-LTV)^2}\sigma_a\sigma_d\rho_{a,d}}$; where: σ_d^2 = the variance of floating-rate debt and $\rho_{a,d}$ = the correlation between asset-level returns and the cost of floating-rate indebtedness.

[14] More broadly, $\frac{\partial k_e}{\partial LTV} = \frac{E(k_a) - r_f - \gamma}{(1-LTV)^2} - 2\delta\frac{LTV}{(1-LTV)^3}$ (*i.e.*, the partial derivative of Equation (3) with respect to the leverage ratio) indicates that, at sufficiently high leverage ratios, the absolute value of the second right-hand term dominates the first right-hand term – thereby producing instances in which additional leverage drives down the expected return on equity.

[15] Given the limited-liability nature of non-recourse financing, the selected parameters should reflect the fact that the loss on equity cannot exceed 100% of invested capital. While this condition may be violated in some of the illustrations that follow, it happens with such *de minimus* probability that it can generally be safely ignored for purposes of these illustrations.

Exhibit 2: Illustration of the Expected Return on Equity and Volatility as a Function of Leverage and Varying Values of θ
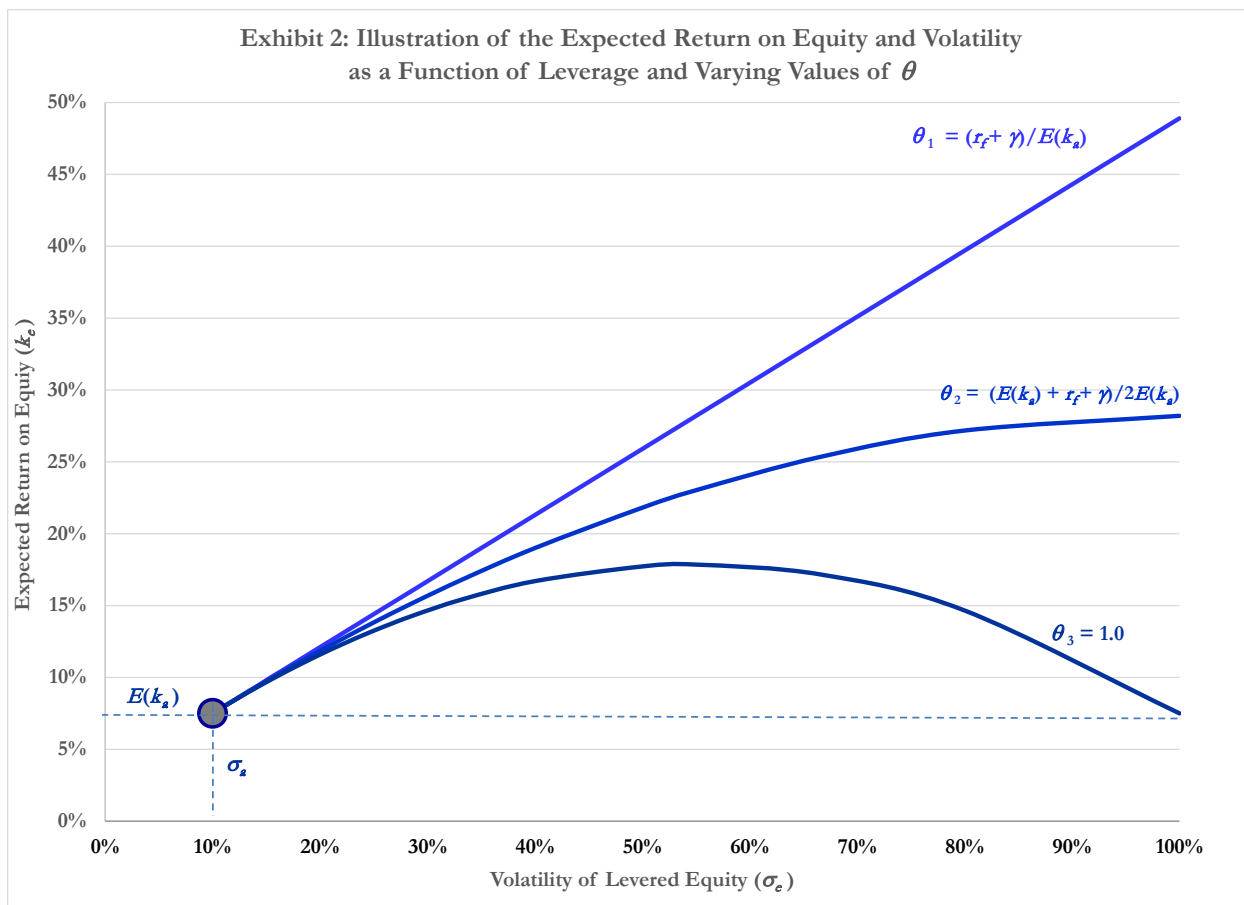
As earlier noted, θ represents the degree to which the lender appropriates the asset's expected return [recall: as $LTV \rightarrow LTV_{Max} \Rightarrow k_d \rightarrow \theta * E(k_a)$]. Said alternatively, the larger the value of θ indicates more costly debt, for a given leverage ratio (*i.e.*, the credit spread rises as θ rises). A few notes about Exhibit 2 are pertinent (for purposes of this discussion, the upper bound of θ is one):

- $\boxed{\theta_1 = \dfrac{r_f + \gamma}{E(k_a)} \Rightarrow k_{d|Max} = r_f + \gamma}$ If the lower limit of θ equals $\dfrac{r_f + \gamma}{E(k_a)}$, then the price of risky

   debt is zero ($\delta = 0$) and, in turn, the cost of indebtedness (k_d) is constant, equal to $r_f + \gamma$, irrespective of the leverage ratio. While this assumption is often made in practice[16] and

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[16] The standard implementation of the corporate finance model is to assume either constant debt costs or some *ad hoc* adjustment to those debt costs. However, in fairness to Modigliani and Miller (1958), they also contemplated (see footnote 17 therein) that: "We can also develop a theory of bond valuation along the lines essentially parallel to those followed for the case of shares. We conjecture that the curve of bond yields as a function of leverage will turn out to be nonlinear…" Consequently, they not only contemplated the pricing of risky debt but, when followed through to its logical conclusion, also the sort of curvilinear risk/return relationship illustrated in Exhibit 2.

thereby generating the familiar linear risk/return continuum, constant borrowing costs (as a function of the leverage ratio) violate the very notion of pricing risky debt.

- $$\boxed{\theta_2 = \frac{E(k_a)+r_f+\gamma}{2[E(k_a)]} \Rightarrow k_{d/Max} = \frac{E(k_a)+r_f+\gamma}{2}}$$ Any value of θ greater than $\frac{r_f+\gamma}{E(k_a)}$

produces a price of risky debt which is greater than zero ($\delta > 0$). When θ is less than or

equal to $\frac{E(k_a)+r_f+\gamma}{2[E(k_a)]}$, the return on levered equity is monotonically increasing, which, in

turn, produces a (concave) curvilinear risk/return continuum, as also implied by McDonald (2006) when examining the expected return on levered equity when risky debt is valued using Black-Scholes option pricing. θ_2 is the largest value of θ that still produces a monotonically increasing return on levered equity (*i.e.*, any value θ greater than θ_2 produces a decline in the expected return on levered equity – for leverage ratios not greater than LTV_{Max}).

- $$\boxed{\theta_3 = 1 \Rightarrow k_{d/Max} = E(k_a)}$$ Any value of θ in excess of $\frac{E(k_a)+r_f+\gamma}{2[E(k_a)]}$ produces, at high

leverage ratios, the irrational outcome that the expected return on levered equity reaches a maximum while the volatility of levered equity continues to increase. It is irrational in the sense that it leads to a condition in which expected levered equity return are declining, while

volatility is increasing. $LTV^* = \frac{E(k_a)-r_f-\gamma}{E(k_a)-r_f-\gamma+2\delta}$ is the leverage ratio at which the expected

return on levered equity is maximized.[17] In the extreme, $\theta_3 = 1$ and the expected return on levered equity equals the expected (unlevered) asset return at the maximum leverage ratio; however, the volatility of the levered equity return equals a multiple (*i.e.*, $1/(1 - LTV_{Max})$) of the volatility of the (unlevered) asset return.[18]

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[17] When $\theta \le \theta_2 = \frac{E(k_a)+r_f+\gamma}{2[E(k_a)]}$, then $LTV^* > LTV_{Max}$, thereby preserving the condition that the expected return

on equity is monotonically increasing. Conversely, when $\theta_2 < \theta \le \theta_3 = 1$, then $LTV^* < LTV_{Max}$, thereby creating a condition whereby that the expected return on equity displays a local maximum. In the special case of $\theta_3 = 1$,

$LTV^* = \frac{LTV_{Max}}{2 - LTV_{Max}} < LTV_{Max}$.

[18] It is also the case that the price (δ) of default risk is twice as expensive when $\theta = 1$ as compared to the limiting case of

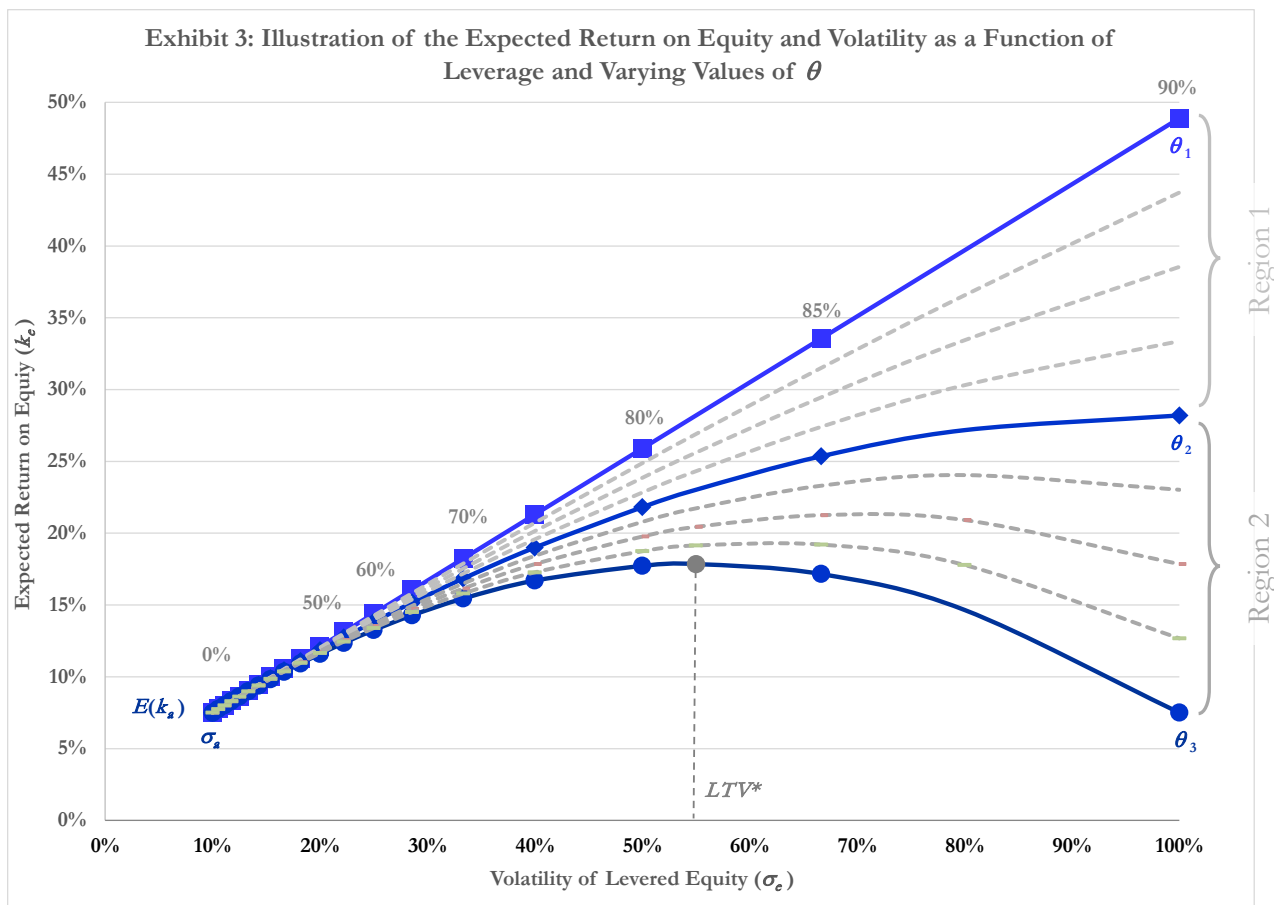when $\theta = \frac{E(k_a)+r_f+\gamma}{2[E(k_a)]}$.

While $\theta > \theta_2$ at leverage ratios greater than LTV^* is irrational from an *ex ante* perspective,[19] it is clearly possible that the expectations about expected asset-level returns $[E(k_a)]$ may differ from the realizations of asset-level returns $\left(\overline{k_a}\right)$. When expected returns exceed realized returns, levered equity returns exhibit the downward sloping curvature shown in Exhibit 2 when leverage ratios are high. In fact, the difference between the expected and realized asset-level returns may be so severe that the debt costs exceed realized asset returns (which implies $\theta > 1$); if so, any amount of borrowing worsens the levered borrower's return (*i.e.*, the borrower experiences "negative leverage"). Finally, it should also be noted that lenders' and borrowers' *ex ante* beliefs about asset-level returns may differ (and it may often be the case that the former is less optimistic than the latter).

Exhibit 3 replicates Exhibit 2 except for three additions which are intended to aid the reader's intuition: *a*) icons corresponding to various leverage ratios have been added, *b*) additional values of θ (and the effect on the risk/return characteristics of levered equity) have been added, and *c*) assuming $\theta_3 = 1$, the leverage ratio which maximizes the return on equity (LTV^*) is identified (given our earlier-indicated parameters, $LTV^* \approx 81.8\%$).

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[19] As an aside, it may be the case that certain investors (typically, non-institutional investors) choose to borrow even when $\theta > \dfrac{E(k_a) + r_f + \gamma}{2\left[E(k_a)\right]}$ at leverage ratios greater than LTV^*. Consider the context of an investor's finite wealth *vis-à-vis* commercial real estate's lack of divisibility and high search costs as well as the unique nature of each building. The confluence of these factors suggests that the specialized knowledge often associated with operating local properties might translate into a particular investor having to utilize more leverage than is optimal in order to acquire a given property, create greater portfolio diversification, *etc.*

Exhibit 3: Illustration of the Expected Return on Equity and Volatility as a Function of Leverage and Varying Values of θ

The addition of identified leverage ratios helps illustrate that, at higher leverage ratios (and, therefore, higher levels of return volatility), the level of θ matters most. At those higher leverage ratios, the expected return on levered equity diverges significantly from using these lower and upper limits of θ. Alternatively stated, there is little difference in the cost of indebtedness for leverage ratios of, say 50% or less of the maximum loan amount and, as such, there is comparatively little difference in the expected return on levered equity. The addition of the grey-dashed lines indicates that there are two regions of possible θ's impact (*i.e.*, the steepness of the credit curve) on levered equity returns. Region 1 constitutes values θ which lie between θ_1 and θ_2, while Region 2 constitutes values θ which lie between θ_2 and θ_3. Any value of θ attributable to Region 1 produces a monotonically increasing expected return on levered equity – irrespective of the leverage ratio (provided that it does not exceed LTV_{Max}). Any value of θ attributable to Region 2 ultimately produces an expected return on levered equity which reaches a maximum – with any additional leverage leading to a decline in the expected return. Finally, the addition of LTV^* (which is the leverage ratio that maximizes the expected return on levered equity) is shown when $\theta_3 = 1$.

III. <u>An Application of the Law of One Price</u>

The law of one price states that two assets which have identical cash flows must have the same price; if not, an arbitrage opportunity exists whereby the underpriced asset is bought and the overpriced asset is simultaneously sold short. In so doing, the arbitrageur locks in a riskless profit. These arbitrage activities drive the convergence in asset prices and markets towards equilibrium prices.

In private markets (real estate or otherwise), shorting a security is typically implausible. Instead, the practical application of the law of one price is that the distribution of likely returns for unlevered "core" (*i.e.*, low-risk/low-return) assets can be transformed through the use of financial leverage to create investments that replicate the distribution of likely returns for unlevered "non-core" assets. If not, the flow of capital from sophisticated investors would favor the underpriced asset and disfavor the overpriced asset, thereby driving market prices towards their equilibrium.

Let's assume that there is investor consensus about the expected return $\left[E\left(k_{\overline{m}}\right) \right]$ and risk $\left(\sigma_{\overline{m}}\right)$ on the "core" market portfolio.[20] More specifically, $E\left(k_{\overline{m}}\right)$ equals the value-weighted average of the expected returns on the market basket of core properties and $\sigma_{\overline{m}}$ equals the value-weighted average of the volatilities of those returns. For simplicity, the same values assumed earlier for $E\left(k_a\right)$ and σ_a are used here for $E\left(k_{\overline{m}}\right)$ and $\sigma_{\overline{m}}$. Note that the latter does not equal the volatility of the return for the market basket, σ_{Mkt}, of core properties.[21]

We then can utilize that consensus view to trace out the equilibrium set of market risk/return opportunities. In principle, arbitrage forces all properties (and investing strategies) to adhere to this continuum. The previous section establishes that, in equilibrium, the levered equity investor – in the presence of risky debt and endogenous borrowing costs – faces a monotonically increasing, but concave, risk/return continuum. As such, Exhibit 4 illustrates the risk/return spectrum on which all

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[20] It is widely viewed that the "core" market (*i.e.*, built and fully leased apartment, industrial, office and retail properties) is substantially larger (in terms of market capitalization) and more homogenous (in terms of pricing, rents, occupancy, tenant quality, *etc.*) than the non-core market (*e.g.*, value-added and opportunistic strategies). Given its larger size and more commodity-like nature, the dispersion of investor opinion concerning the core market is likely to be narrower than for the non-core market.

[21] Technically, $E\left(k_{\overline{m}}\right) = \sum_{i=1}^{N} w_i E\left(k_{a,i|Core}\right)$; where: w_i = the proportion of each property allocated to the ith property, for all N properties. And, $\sigma_{\overline{m}} = \sum_{i=1}^{N} w_i \sigma_{a,i|Core}$, while $\sigma_{Mkt} = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \rho_{i,j} \sigma_{a,i|Core} \sigma_{a,j|Core}}$. As such, $\sigma_{\overline{m}} > \sigma_{Mkt}$ when $\overline{\rho_{i,j}} < 1$.

unlevered asset strategies[22] should rest in equilibrium and, ultimately, the continuum also contains all levered strategies as well.

Some assets will have less expected return and risk than the (core) market portfolio (estimated by $E\left(k_{\overline{m}}\right)$ and $\sigma_{\overline{m}}$).[23] To a point, these assets can be thought of as properties that are less volatile (*e.g.*, an office building leased on a long-term, net basis to a federal agency) than average core property; it is important to understand how to also price these lower-risk assets. However and as we approach the risk-free rate, it is perhaps more intuitive to think of these as investments in various debt securities (starting with high-yield debt and then proceeding to first mortgages) – among others, see Pagliari (2017). That is, though this approach began with the market portfolio and, through the use of leverage, constructed the risk/return continuum for higher-risk/higher-return assets, the approach also provides a method by which the risk/return continuum extends to lower-risk/lower-return assets.

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[22] The ranges of unlevered value-added and opportunistic returns are provided merely as an illustration. In practice, there tends to be much imprecision and disagreement as to the risk/return characteristics of the non-core assets.

[23] Here, *LTV* ratios are negative; the absolute value of which represents the portion invested in the risk-free asset. In turn, the volatility function (Equation (4)) was modified to acknowledge a portion of the portfolio being invested in the risk-free asset: $\sigma_e = \left(1 - \left|LTV\right|\right)\sigma_a$. The expected returns were determined using Equation (5).

Exhibit 4: Illustration of Market's Equilibrium Risk/Return Continuum
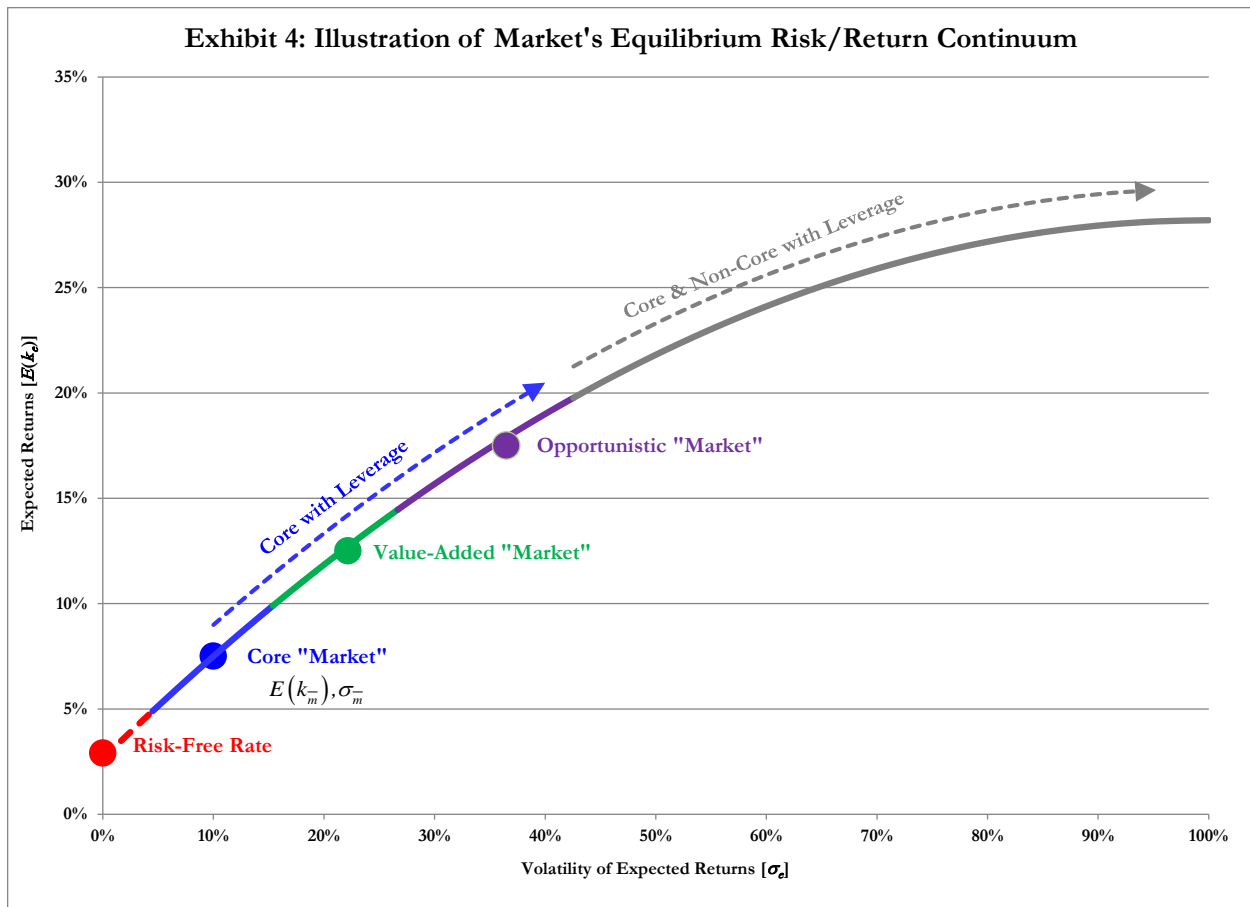
All real estate investors have the practical (as opposed to the solely theoretical) opportunity of simply leveraging up their core assets as an alternative to investing in riskier non-core assets. In equilibrium, investors' funds will flow to whichever alternative provides the greater expected return after holding risk constant (or *vice versa*), thereby eliminating, merely via financial engineering, any arbitrage possibilities (as the price of the overvalued alternative falls and the price of the undervalued alternative rises). For the integrity of the law of one price to be maintained, it must also be the case that both unlevered and levered investment opportunities lie on this continuum. As one example, a core property with significant leverage should replicate the risk/return characteristics of a non-core property with modest leverage.

However, this depiction of the risk/return continuum differs from the standard finance literature in two important respects: First (and as noted earlier), this paper argues for a curvilinear risk/return continuum, which stands in stark contrast to the linear relationship most often asserted (*e.g.*, see Campbell (2018)). The standard finance literature imagines a frictionless market in which investors, among other things, can costlessly short any security. While the public market might closely approximate these requirements,[24] the same cannot be said about the private market. And

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[24] Schliefer and Vishny (1997) and Lamont and Thaler (2003) point out some of the limits of these arbitrage arguments.

accordingly, it is argued here that the risk/return continuum is curvilinear. Interestingly, several authors – including Fama (1986), Fama and French (1992, 2004) and Frazzini and Pedersen (2014) – find that the realized risk/return continuum for publicly traded securities (for debt instruments as well as equities) is flatter than implied by the standard finance literature. While those papers look for additional (systematic and priced) factors[25] to explain this result, this paper argues that the flattening of the risk/return continuum is a natural outgrowth of considering risky debt.

Second, this depiction also differs from the standard finance literature in that total risk (σ_a) is considered, rather than some measure of undiversifiable or systematic risk. Given the asset-specific and asymmetric nature of the lender's participation in the asset's total return, total risk is clearly the metric by which lenders ought to price default risk (*i.e.*, the value of the (non-recourse) borrower's put option is a function of total risk (σ_a) – not merely systematic risk). However, the same argument cannot be made for levered equity investors. For such investors in the public market, it is widely assumed that idiosyncratic or unsystematic risk can be costlessly diversified away and, in turn, it is only systematic risk that should be priced. The (one-factor) capital asset pricing model (CAPM) of Sharpe (1964) and others posits, among other features, that all investors hold the "market" portfolio (and that there are no non-traded assets); in such a world, systematic risk is measured by the security's covariance with the broader market – more specifically, the security's "beta" (β). Clearly, private-market assets, like commercial real estate, do not comport well with CAPM's underlying assumptions – including: no private investor can own the market portfolio. As such, private real estate investors are concerned with total risk.

That said, it is of course natural to assume that at least some private real estate investors are also concerned with how individual properties contribute to the risk of their (commercial real estate) portfolio.[26] Despite the private real estate investor's inability to own the market portfolio, it is helpful to recall CAPM's basic tenets about pricing systematic risk via a security's beta, which – for the ith security – is $\beta_i = \rho_{i,Mkt} \, \sigma_i / \sigma_{Mkt}$; where: $\rho_{i,Mkt}$ = the correlation of the ith security's return to the market's return, σ_i = the standard deviation of the ith security's return, and σ_{Mkt} = the standard deviation of the market's return. Since σ_{Mkt} is merely a scalar, the important components of beta are $\rho_{i,Mkt}$ and σ_i. In this paper's setup, σ_i is equivalent to σ_a; so, the only unaccounted component is $\rho_{i,Mkt}$. Assuming that investors are also concerned with this aspect of diversification,[27] then several

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[25] Interestingly, Frazzini and Pedersen (2014) argue that one of these factors is "betting against beta" whereby low-beta securities are held long and high-beta securities are sold short.

[26] However, for some property investors, diversification is less important than focus. The public real estate market provides a natural experiment for comparing the effects of diversification versus concentration; among others, see: Ro and Ziobrowski (2011).

[27] This implies that investors believe that there is significant variation in $\rho_{i,Mkt}$ across possible property investments; in the alternative, if investors believe that there is insignificant variation in $\rho_{i,Mkt}$ across properties, then $\rho_{i,Mkt}$ is tantamount

possibilities exist including: *a*) investors utilize crude metrics of property type, geography and/or other "style" characteristics to naively diversify their portfolios (*i.e.*, a form of constrained portfolio optimization) or *b*) due to the fact that private markets do not comport well with the assumptions underlying CAPM, investors separately estimate each of $\rho_{i,j}$ and σ_i for each of the property investments in their portfolios.[28] In the latter instance, investors may consider a three-dimensional surface in which expected returns are an explicit function of $\rho_{i,j}$ and σ_i – as opposed to collapsing these two elements into one (β_i).

Lastly and given the breadth of both the private and public real estate markets, it cannot be the case that pricing of risk in one market substantively differs from the pricing in the other market. Ultimately, the pricing practices of the private and public markets must converge with one another; otherwise, a significant arbitrage opportunity would persist. Instead, we empirically observe (see Pagliari, *et al.* (2003) and others) that the long-run returns of these two markets – after controlling for leverage (and other factors) – are statistically indistinguishable from one another, thereby eliminating persistent arbitrage opportunities.

IV. <u>Delegated Investment Management</u>

Let's next consider delegated investment management. In those instances where passive real estate investors (*e.g.*, limited partners) invest a portion of their wealth in a particular property or fund, there is typically a contingent-profits interest provided to the operating (or general) partner (which, in turn, provides expertise – including the sourcing, financing and operation of the property or fund). This residual-profits interest creates an asymmetric, option-like participation for the operating partner and well-known principal/agent problems (*e.g.*, see Pagliari (2015) among others). Like any contingent claim, the value of the option increases with the volatility of the underlying security. Accordingly, the operating partner may be perversely motivated to leverage the property in excess of *LTV** because the expected value of its promoted interest increases with this leverage-induced volatility.

Let's revisit the earlier illustration of levered equity returns (Exhibit 3) and examine the net return to limited partners when the general partner receives a promoted (or carried) interest. For purposes of illustration, let's assume that the limited partners receive a preferred return of 7.5% per annum and

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to yet another scalar (with σ_{Mkt} acting as the other a scalar). Real estate investing differs from stock investing more broadly, in which the latter permits investing across industries (*e.g.*, energy, financials, tech, utilities, *etc.*) – whereas real estate is only one industry.

[28] For a given portfolio, Fama (1976) indicates that this contribution consists of the proportion of the portfolio invested in that asset and the weighted average of the pairwise co-variances between the returns of that asset and all other assets in the portfolio.

that the general partner receives 20% of any excess profits.[29] (To help illustrate the "optionality" embedded in the general partner's promoted interest, the limited partner's preferred return was specifically set equal to the (unlevered) asset's/fund's expected return – a point we will revisit.)

1. **An Equilibrium Case:** θ_2

Of the potential equilibrium cases $\left[\dfrac{r_f + \gamma}{E(k_a)} < \theta \leq \dfrac{E(k_a) + r_f + \gamma}{2\left[E(k_a)\right]} \right]$, let's examine the one in which
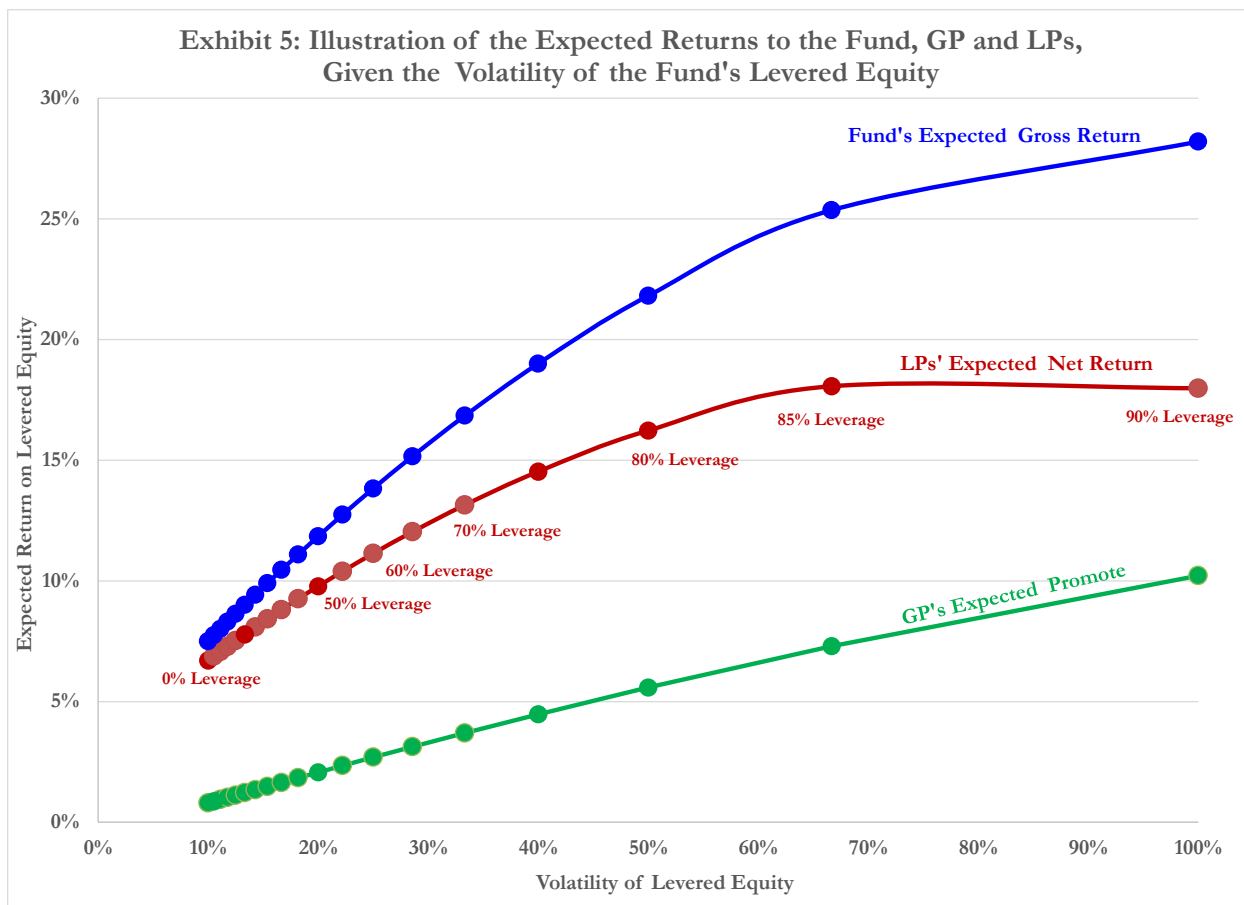
the lender extracts the largest surplus: $\theta_2 \left[= \dfrac{E(k_a) + r_f + \gamma}{2\left[E(k_a)\right]} \right]$. In this case (and given our earlier

assumptions), the returns to levered equity as well as to each of the general and limited partners are illustrated in Exhibit 5: [30]

Exhibit 5: Illustration of the Expected Returns to the Fund, GP and LPs, Given the Volatility of the Fund's Levered Equity

The upper (blue) line represents the fund's gross return across various leverage ratios, using the same assumptions as Exhibits 3 and 4. Because of varying fee structures across fund offerings, a fund's gross return is often quoted when discussing performance, evaluating track records, *etc.* The lower (green) line represents the expected amount of the fund's gross return to be paid to general partner due to its promoted interest.[31] (The calculations determining the expected promoted interest are provided in Appendix B.) It is clear that the value of the expected "promote" increases linearly[32] with the volatility of the fund's return (*e.g.*, as more leverage and/or riskier properties are added to the fund.) The middle (maroon) line represents the investors' net return. Given the concavity of the fund's returns and the linearity of the general partner's promoted interest, the concavity of the investor's (net) return is more pronounced than the fund's return. And even though the fund's expected returns are monotonically increasing, there is a local maximum with regard to the investors'
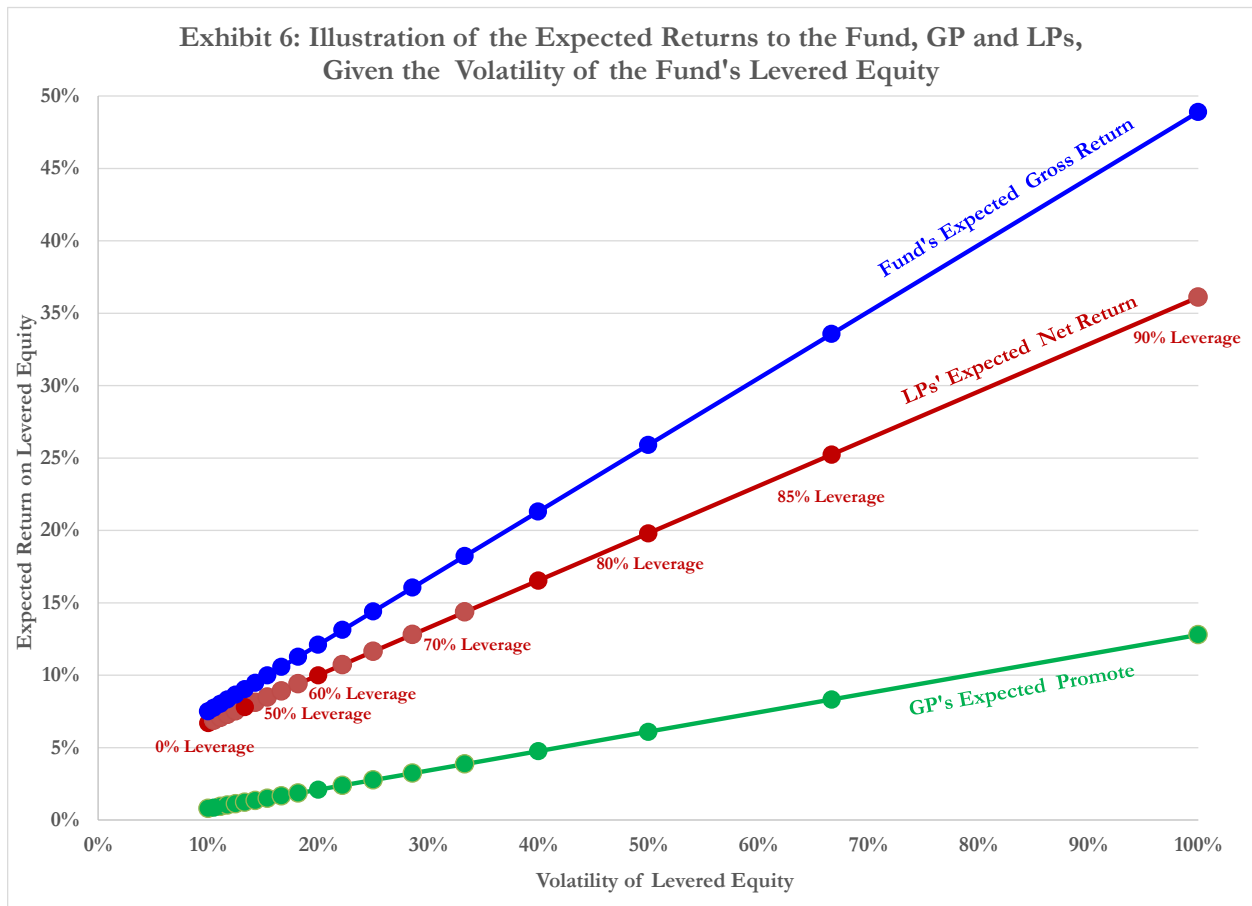
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[31] For simplicity, any base fees additionally paid to the general partner have been ignored. While it is often argued that these base fees merely "keep the lights on" for the general partner, they unquestionably lower the investors' net return – regardless of the fund's (or venture's) profitability. This being true, including base fees would not change the curvature of the investors' net return.

[32] There is a small exception at the highest leverage ratios (at which, the concavity of the fund's return is most pronounced); at these higher leverage ratios, the slope of the general partner's expected promote (*vis-à-vis* the volatility of the fund's return) declines somewhat – almost imperceptibly so, given the scale of Exhibit 5.

net return; at which point, further borrowing is irrational for the limited partners, but not necessarily so for the general partner. Even if there were no local maximum, the flattening of the investors' net returns at high leverage ratios would require very low levels of risk aversion in order for the limited partners to find these highly levered (net) returns to be attractive. Notwithstanding that the expected value of the general partner's promoted interest is (linearly) increasing with the volatility of the fund's return, the general partner rationally views the riskiness of that return in light of the firm's likely future fundraising efforts – where poor (realized) performance can greatly reduce the success of those future efforts (*e.g.*, see Panageas and Westerfield (2009)).

2. **A Non-Equilibrium Case**: $\underline{\theta}_1$

Let's next examine one of the two limiting (albeit, irrational) cases. This first case considers instances in which borrowing costs are constant across all leverage ratios: $\underline{\theta}_1$ (*i.e.*, $\underline{\theta}_1 \Rightarrow \delta = 0$). While this case (as noted earlier) violates the notion of risky debt, it is instructive to inspect the payoffs to the general and limited partners in such an environment – see Exhibit 6:

Exhibit 6: Illustration of the Expected Returns to the Fund, GP and LPs, Given the Volatility of the Fund's Levered Equity
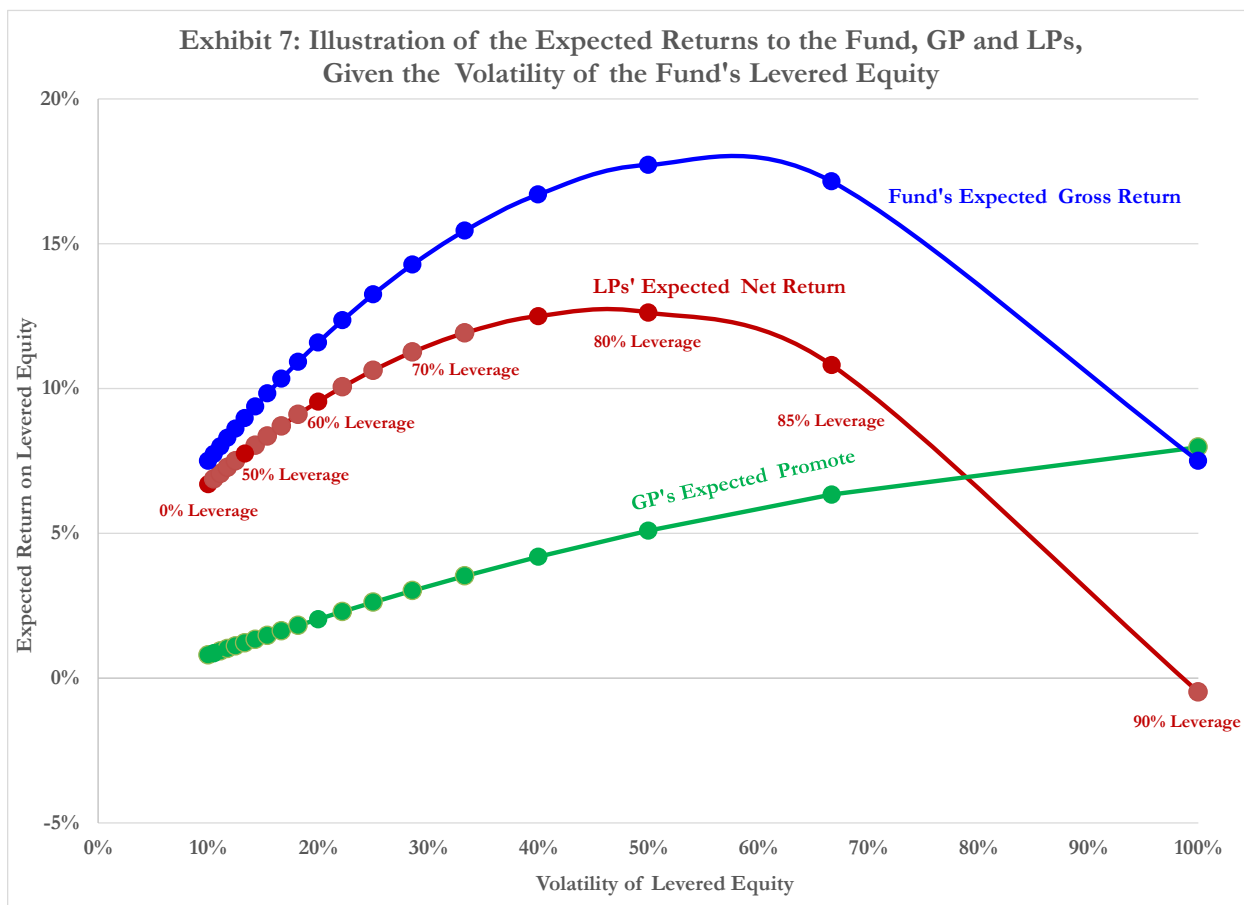
As in the equilibrium case (see Exhibit 5), the expected return with respect to the general partner's promoted interest is linear (or nearly so) with regard to the volatility of the fund's return. But

because the fund's return is also linear, so is the nature of the limited partner's return (*i.e.*, it too is linear with regard to the volatility of fund-level returns). In fact, this is the only case in the limited partner's return is linear with regard to fund-level volatility. Unfortunately for limited partners, this outcome is fictional, as borrowing at the riskless rate is unavailable. (In all other cases, the limited partner's return is concave with respect to fund-level volatility.) Nevertheless, the difference between the fund's return and the limited partner's return is widening as leverage/volatility increases – because the dilution of the limited partners' expected (net) return, due to the general partner's expected promote, is increasing with the leverage ratio.

3. **A Non-Equilibrium Case**: θ_3

Finally, let's examine the second of the two limiting (albeit, irrational) cases. This second case considers those instances in which borrowing costs equal, at the maximum leverage ratio, the asset's expected return (*i.e.*, $\theta_3 = 1$) – see Exhibit 7:

Exhibit 7: Illustration of the Expected Returns to the Fund, GP and LPs, Given the Volatility of the Fund's Levered Equity

As indicated in Exhibit 7, the concavity of both the fund's and the limited partner's return are quite pronounced (so much so, as earlier noted, that the fund's expected levered return equals the fund's expected (unlevered) asset return at the maximum leverage ratio (LTV_{Max}); there, the general

partner's expected promoted interest slightly exceeds the fund's expected return and the limited partner's expected (net) return is slightly negative. And, given the extreme concavity of the fund's expected return (as a function of leverage/volatility), it is unsurprising that the concavity of the general partner's expected promoted interest as illustrated above is the most extreme of the three cases examined here.

4. **A Précis**:

When examining investors' (net) returns in light of risky debt it is apparent that, even though the fund's expected returns are monotonically increasing, there may be a local maximum with regard to the investors' net return; at which point, further borrowing is irrational for the limited partners (even if they are risk-neutral), but not necessarily so for the general partner. Even if there is no local maximum, the flattening of the investors' net returns at high leverage ratios requires very low levels of risk aversion on the part of limited partners in order to find these highly levered (net) returns to be attractive. All of this begs several (unanswered) questions:

> Given the typically immense asymmetric information divide as between the principals (*i.e.*, the limited partners) and the agents (*i.e.*, the general partners), is it reasonable to believe that limited partners can even observe the significant dilution in their expected returns for those fund utilizing significant leverage (say, more than 75% in our illustrations)? [33]

> If not, do limited partners rely on the general partner to set reasonable leverage ratios (as both the limited and general partners know that the riskiness of highly levered returns might jeopardize the success of future fundraising efforts)?

> If not, do limited partners set *ad hoc* leverage limits on the funds in which they invest?

Additionally, note that, in all three cases illustrated here, the increase in the fund's expected return is solely due to financial engineering (*i.e.*, it is assumed that that the fund acquires core assets, at market prices, and levers them up (or, equivalently, acquires – with less leverage – non-core assets at market prices)); yet, the expected value of the general partner's promoted interest is increasing with the degree of leverage. In these illustrations, the general partner has displayed no "skill." The empirical evidence, with regard to the risk-adjusted (net-of-fees) performance of non-core funds, is not encouraging – see Bollinger and Pagliari (2019) and Pagliari (2016) – which potentially suggests that even many institutional investors may have not fully discerned these lessons about the interplay of risky debt, leverage and delegated investment management fees.

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[33] These high leverage ratios may be particularly emblematic of the opportunity funds. While their "targeted" leverage ratios may be stated to be at or below 75%, their use of "subscription" lines (*i.e.*, borrowing against committed, but not yet contributed, capital of the limited partners) may increase the fund's effective leverage ratio substantially above such targets. For example, see: Marks (2017).

V. <u>Concluding Remarks</u>

The prevalence of financial leverage in private, commercial real estate investing indicates that investors are well-served to understand the impacts of such leverage on their investments – particularly given the disappointing performance many institutional investors have experienced with many of their high-risk/high-return (*i.e.*, high-leverage) funds. Using a simple reduced-form model to approximate the pricing of risky debt, this article focuses on three aspects of levered equity returns when investing in private commercial real estate: First, the positive relationship between the project's leverage ratio and the costs of mortgage-loan borrowing creates a concave risk/return continuum for levered equity investors. The lender's pricing of risky debt is such that the equilibrium return on levered equity asymptotically reaches a maximum; while additional borrowing increases the expected return on equity, it only marginally does so. Second, the law of one price suggests that this curvilinear continuum represents the basis on which higher-risk/higher-return investments ought to be measured – as opposed to the more widely theorized linear relationship (which assumes constant borrowing costs). Third and when considering delegated investment management, additional borrowing linearly increases the expected value of the sponsor's (or general partner's) promoted interest, which directly reduces the expected return of the investor's (or limited partner's) net return. In almost all cases, this reduction in the investor's (net) return leads to an optimal leverage ratio well below the maximum available. All three of these impacts are attenuated at the highest leverage ratios.

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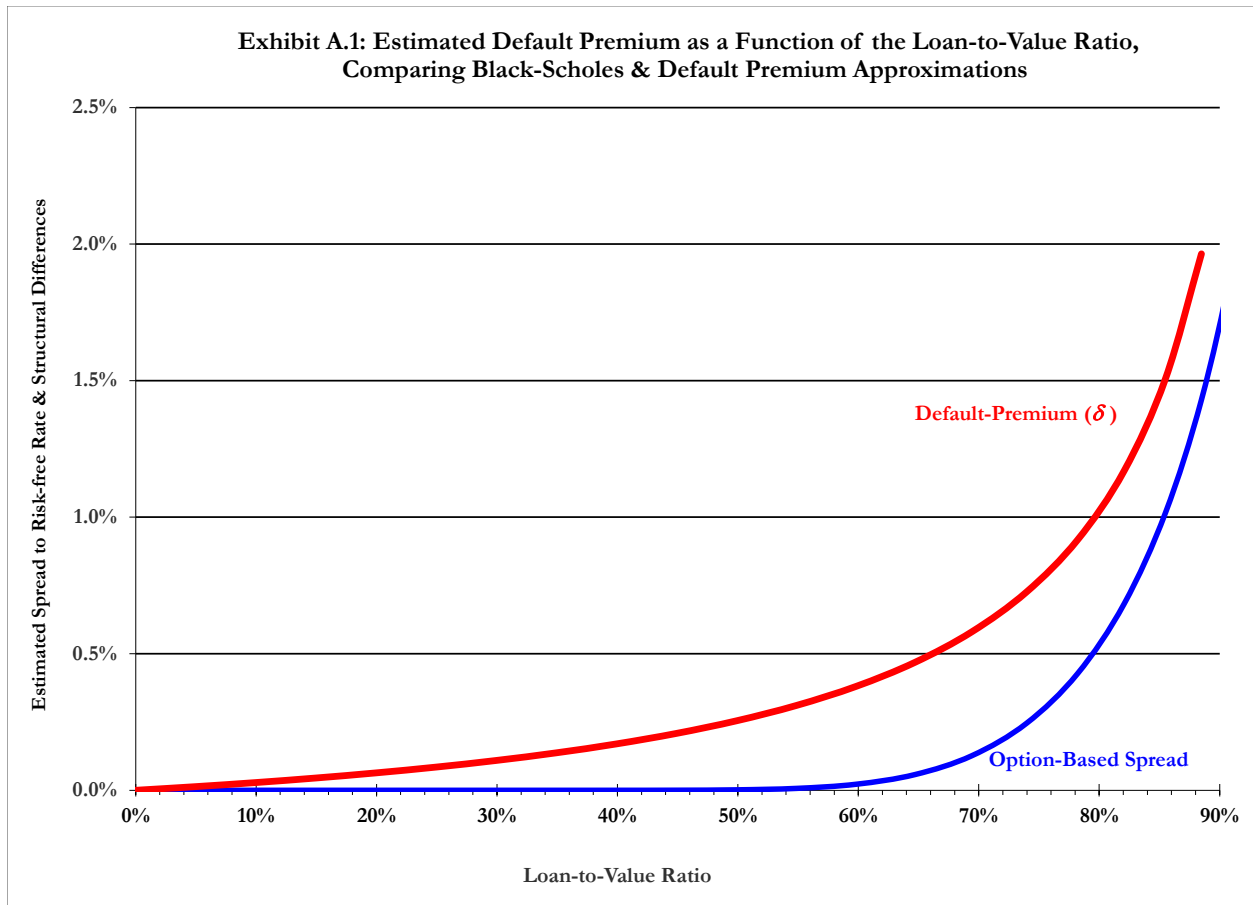
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VII.   Appendix A

As a first attempt at understanding the theoretical basis for estimating the increase in the lender's interest rate as the loan-to-value increases, assume a simple application of the Black-Scholes (1973) option-pricing model in which the underlying asset pays no dividend. In so doing, we can view the lender as proving a riskless loan less the value of the borrower's put option – given a particular commercial loan obligation. To illustrate the example, let's begin with our earlier assumption about the adjusted risk-free rate (*i.e.*, $r_f + \gamma = 2.9\%$) and asset-level volatility (*i.e.*, $\sigma_a = 10\%$); moreover, let's further assume that the term of the mortgage loan equals 5 years.[34]

Exhibit A.1 compares the option-based pricing of the default premium to a rational value of θ with respect to the default-premium (δ) approximation:

Exhibit A.1: Estimated Default Premium as a Function of the Loan-to-Value Ratio, Comparing Black-Scholes & Default Premium Approximations

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[34] While the term to maturity (T) is immaterial (assuming σ_a is time-invariant) with respect to the calculation of the estimated default premium (δ), it can matter a great deal to the valuation of the put option (*i.e.*, as T increases, the value of the option increases). The choice of $T = 5$ is arbitrary.

The default-premium approximation is based on the largest return that the lender can demand in

equilibrium: $\theta_2 \left[= \dfrac{E(k_a) + r_f + \gamma}{2[E(k_a)]} \right]$. In this instance, the default-premium approximation (*i.e.*, the red

line) is always above the option-based pricing of the default premium (*i.e.*, the blue line), with the difference averaging approximately 25 basis points over the range examined above: $0 \leq LTV \leq LTV_{Max} = 90\%$ (by assumption).[35] The magnitude aside, this result is consistent with Culp, Nowaza and Veronesi (2018).

If one is to believe that the option-pricing methodology is superior for our purposes (despite the earlier criticisms), then it is apparent that the default-premium approximation overstates the option-based default premium at all relevant loan-to-value ratios. The impact of these differences in the expected debt cost is that default-premium approximation produces an understatement of the expected return (for a given level of equity volatility).

In this paper, the default-premium (δ) approximation is utilized and the curvature is determined by $\dfrac{LTV}{1-LTV}$. As another perspective, consider the hyperbolic depreciation rates (*e.g.*, see Hoteling (1925)) to model different curvatures. Assuming the salvage value equals zero and the asset's useful life equals T, the value of the asset at time t is scaled by: $V(t) = \dfrac{T-t}{T-\phi t}$, where: ϕ = the rate of obsolescence (or curvature). The hyperbolic depreciation function captures the change in asset values over time. For our purposes, we are interested in values of $0 < \phi < 1$, such that the function is concave (alternatively, $\phi = 0 \rightarrow$ linear and $\phi < 0 \rightarrow$ convex) and utilizing $1 - V(t)$ as the credit-pricing function. Applying this function to range of leverage ratios from zero to LTV_{Max} and recognizing the minimum debt cost equals $r_f + \gamma$ and maximum debt cost equals $\theta E(k_a)$ produces the appropriate convex credit spread. Moreover, $\phi = .9$ exactly replicates the curvature of $\dfrac{LTV}{1-LTV}$;

as such, please see Exhibit A.2 for an illustration:

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[35] At least theoretically, the option-pricing approach may consider leverage ratios above LTV_{Max} (but below 100%).

**Exihibit A.2: Illustration of the Cost of Indebtednss
as a Hyperbolic Function of the Leverage Ratio**
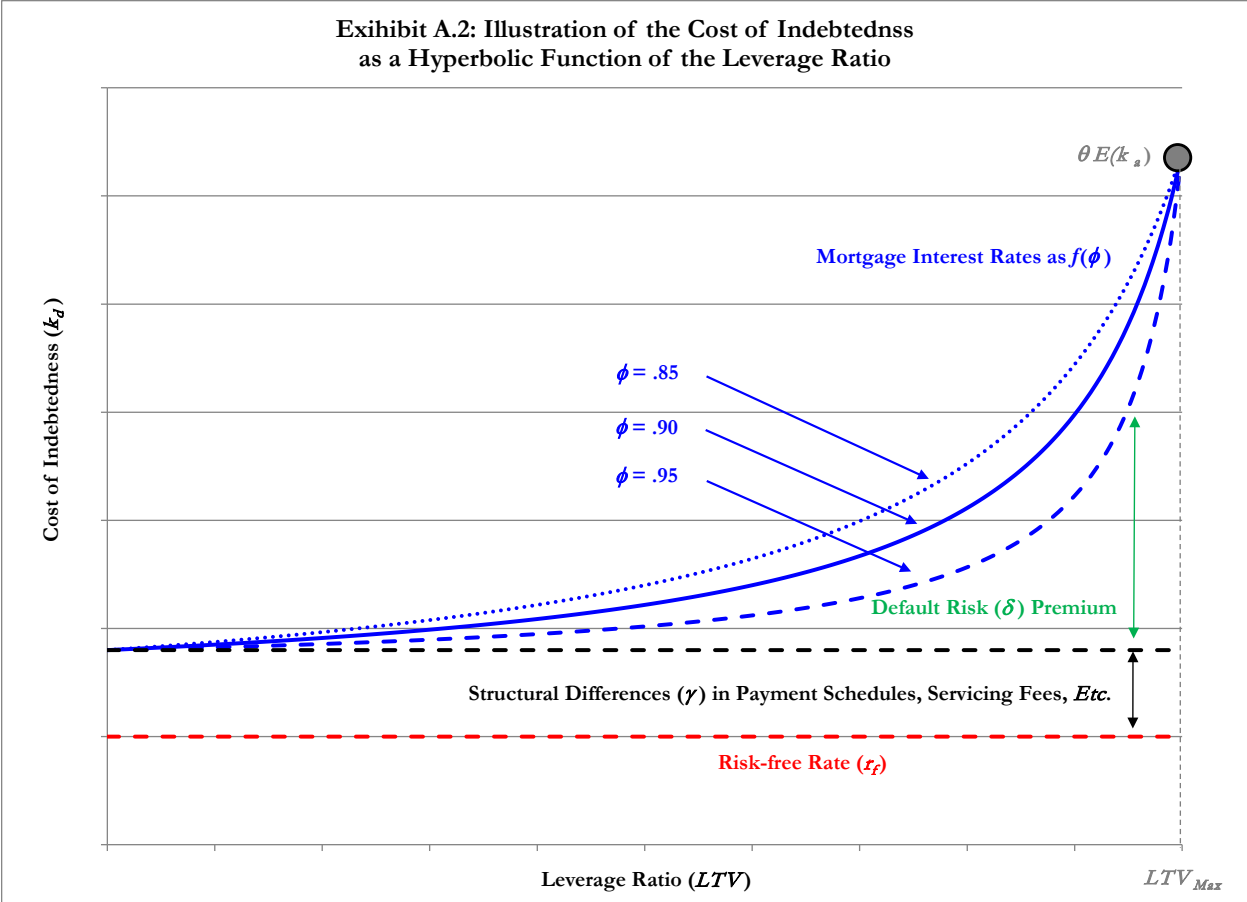
However, this discussion about discrepancies in the curvatures of various estimates of pricing risky debt masks the important point: In all (equilibrium) cases, the existence of risky debt produces a curvilinear risk/return continuum. Indeed, the exact curvature of risky-debt pricing is not our immediate purpose; instead, the concavity of the risk/return continuum is produced whenever the price of risky debt is positive ($\delta > 0$) and its impact on both the pricing of commercial properties and the limited partner's expected return is attenuated.

VIII. Appendix B

The expected value of the manager's promoted interest, $E[\pi]$, can be given by (see Pagliari (2007)):

$$E[\pi] = \kappa \int_{\psi}^{\infty} f(x)(x - \psi) dx \tag{B.1}$$

where: ψ = the limited partners' preferred return and κ = the general partner's promoted interest (stated as a percentage of excess profits). Assuming the fund's returns are normally distributed, $\left[x_i \sim N\left(E(k_e), \sigma_e^2\right) \right]$, any normal distribution which is truncated beginning at ψ has a conditional

mean of $E[x \mid x \geq \psi] = \mu + \sigma \lambda(\alpha)$ where: $\delta(\alpha) = \lambda(\alpha)(\lambda(\alpha) - \alpha), \lambda(\alpha) = \dfrac{\phi(\alpha)}{1 - \Phi(\alpha)}$,

$\alpha = \dfrac{\psi - E(k_e)}{\sigma_e}$, $\phi(\alpha)$ = the probability density function of α, and $\Phi(\alpha)$ = the cumulative

distribution function of α – see Greene (2011).[36]

Accordingly, the mean of a truncated normal distribution is given as a restatement of Equation (B.1) and by Equation (B.2):
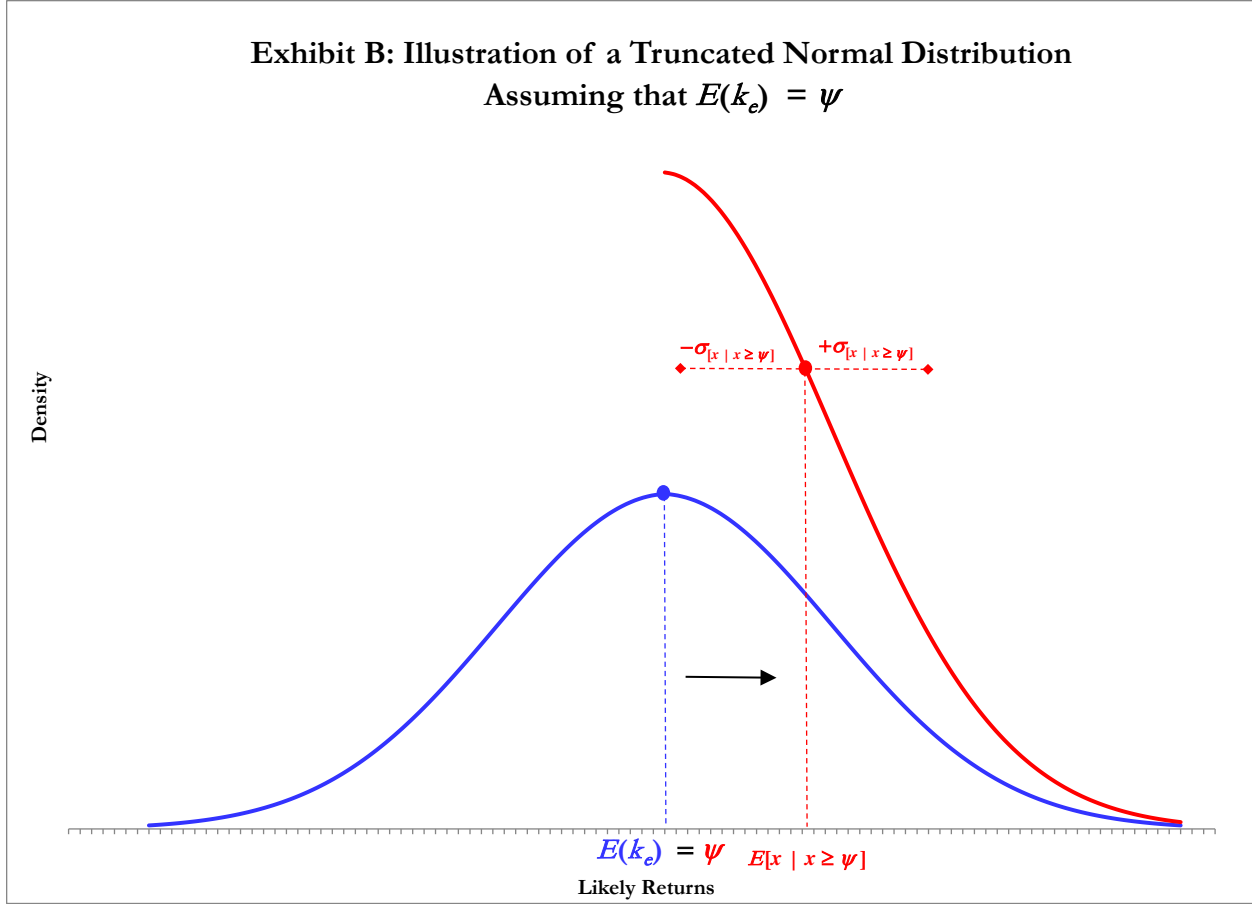
$$\int_{a=\psi}^{\infty} (x) f(x / x > a) dx = E[x / x > a] = E(k_e) + \sigma_e \frac{\phi(\alpha)}{1 - \Phi(\alpha)} \tag{B.2}$$

One particular view, when $E(k_e) = \psi$, of the truncated normal distribution looks like that illustrated in Exhibit B:

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[36] I thank Greg MacKinnon for referring this citation to me.

Exhibit B: Illustration of a Truncated Normal Distribution
Assuming that $E(k_e) = \psi$

By extension, the investing partner's expected net return, $E(\eta)$, equals the venture's expected return less the expected value of the general partner's promoted interest:

$$E(\eta) = E(k_e) - E[\pi] \tag{B.3}$$

Or, equivalently, the limited partner's expected net return, $E(\eta)$, can also be viewed as the weighted sum of the expectations of the results that occur when the fund's's return is less than and greater than the preferred return (ψ):

$$E(\eta) = E[x/x < a]\Phi(\alpha) + (1-\kappa)\left[E[x/x > a] - \psi\right](1 - \Phi(\alpha)) \tag{B.4}$$

$$= \left[E(k_e) - \sigma_e \frac{\phi(\alpha)}{\Phi(\alpha)}\right]\Phi(\alpha) + (1-\kappa)\left[E(k_e) + \sigma_e \frac{\phi(\alpha)}{1 - \Phi(\alpha)} - \psi\right](1 - \Phi(\alpha))$$